Topological Quantum Error Correction: The Toric Code

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Nicolai Lang

Institute for Theoretical Physics III
University of Stuttgart
Outline

1. Introduction
   → (Quantum) Error correction
   → The Stabilizer formalism

2. The Toric Code
   → Rigorous construction
   → Error detection & correction
   → Experimental results
The big picture

Topological Quantum Memories & Computing

Quantum-information-theory

Algebraic Topology (...)

IGT (7th floor)

PI 3 (6th floor)

You

Toric Code

Me

Condensed Matter Physics

ITP 3 (5th floor)
Introduction
Quantum error correction and the Stabilizer formalism
(Quantum) Error correction

General idea:
→ Distribute few logical (qu)bits onto many physical (qu)bits
→ Hope: Errors do not "reach" logical (qu)bits
(Quantum) Error correction

General idea:
- Distribute **few logical** (qu)bits onto **many physical** (qu)bits
- Hope: Errors do not "reach" logical (qu)bits

Classical

- Clone bits 😊
- Arbitrary measurements 😊

("Repetition Code")

Quantum

- No-cloning Theorem 😞
- Restricted measurements 😞
Solution: Quantum Codes & Quantum Error Correction

"The first one" (Shor Code, 1995)

"The exciting one" (Toric Code, 1997)

Topological QEC

Self-Correction

(Quantum) Error correction

Solution: Quantum Codes & Quantum Error Correction

"The exciting one"

(Toric Code, 1997)

Topological QEC

The stabilizer formalism

**Goal:**
→ Describe highly entangled many-qubit states

**Problem:**
→ Exponentially large Hilbert space:

\[ N \text{ qubits} \quad \rightarrow \quad \mathcal{H} = \bigotimes_{i=1}^{N} \mathbb{C}^2 \quad \rightarrow \quad \dim \mathcal{H}^N = 2^N \]

**Solution:**
→ Use group theoretical methods
→ Describe **stabilizer group** instead of the **stabilized state**
The stabilizer formalism

Pauli Group

\[ G_N := \text{span}\left\{ 1_1 \otimes \cdots \otimes \sigma_k^i \cdots \otimes 1_N \mid i \in \{x, y, z\}; 1 \leq k \leq N, k \in \mathbb{N} \right\} \]
The stabilizer formalism

**Pauli Group**

\[ G_N := \text{span} \left\{ 1_1 \otimes \cdots \otimes 1_N \otimes \sigma_k^i \otimes 1_N \mid i \in \{x,y,z\}; 1 \leq k \leq N, k \in \mathbb{N} \right\} \]

**Stabilizer**

Choose Generator: \[ G = \{ g_i \}_{i=1}^d \subseteq G_N \]

Define: \[ S := \text{span} G \quad \text{and} \quad \mathcal{P}S := \left\{ |\Phi\rangle \in \mathcal{H}^N \mid S |\Phi\rangle = |\Phi\rangle \right\} \]

Independent & commuting generators

Stabilizer (group) \quad \text{Stabilized subspace (state)}
The stabilizer formalism

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Stabilizer

Choose Generator: \( G = \{g_i\}_{i=1}^d \subseteq G_N \)

Define: \( S := \text{span } G \) and \( \mathcal{P}S := \{ |\Phi\rangle \in \mathcal{H}_N \mid S |\Phi\rangle = |\Phi\rangle \} \)

Dimension & Rank

\[ \mathcal{P}S \leq \mathcal{H}_N \quad \& \quad d = \text{rank } S \quad \Rightarrow \quad \dim \mathcal{P}S = 2^{N-d} \]
The stabilizer formalism

**Pauli Group**

\[ G_N := \text{span} \left\{ 1 \otimes \cdots \otimes \sigma_k^i \cdots \otimes 1_N \mid i \in \{x,y,z\}; 1 \leq k \leq N, k \in \mathbb{N} \right\} \]

**Stabilizer**

Choose Generator: \( \mathcal{G} = \{ g_i \}_{i=1}^d \subseteq G_N \)

Define: \( S := \text{span} \mathcal{G} \) and \( \mathcal{PS} := \{ |\Phi\rangle \in \mathcal{H}^N \mid S |\Phi\rangle = |\Phi\rangle \} \)

**Dimension & Rank**

\[ \mathcal{PS} \leq \mathcal{H}^N \quad \& \quad d = \text{rank} S \quad \Rightarrow \quad \dim \mathcal{PS} = 2^{N-d} \]

→ \( S \) abelian subgroup of \( G_N \)
→ Describe \( \mathcal{PS} \) in terms of \( \mathcal{G} \)
→ Not all states are stabilizer states

**Example:**

\( N = 2 \) \quad \mathcal{G} = \{ \sigma_1^x \sigma_2^x, \sigma_1^z \sigma_2^z \} \subseteq G_2 \)

\( d = 2 \)

\[ [\sigma_1^x \sigma_2^x, \sigma_1^z \sigma_2^z] = 0 \]

\[ \dim \mathcal{PS} = 2^{2-2} = 1 \]

\[ \mathcal{PS} = \left\{ |\text{EPR}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \right\} \]
2
The Toric Code
A rigorous introduction
The Setting

1. Define the topology

Square lattice with periodic boundary conditions embedded into the torus:

The lattice defines \( |V(\mathcal{L})| = L^2 \) vertices and \( |E(\mathcal{L})| = 2L^2 \) edges.

The embedding defines \( |\mathcal{F}(\mathcal{L})| = L^2 \) faces/plaquettes.
The Stabilizer
2. Define the Hilbert space
Attach spins/qubits to all edges:

$$\mathcal{C}_e^2, e \in E(\mathcal{L}) \rightarrow \mathcal{H}(\mathcal{L}) = \bigotimes_{e \in E(\mathcal{L})} \mathcal{C}_e^2$$
2. Define the Hilbert space
Attach spins/qubits to all edges:
\[ C_e^2, e \in E(\mathcal{L}) \rightarrow \mathcal{H}(\mathcal{L}) = \bigotimes_{e \in E(\mathcal{L})} C_e^2 \]

3. Define the stabilizer
A star operator per vertex, ...
\[ A_s := \prod_{i \in s} \sigma_i^x \in \mathcal{V}(\mathcal{L}) \]
... a plaquette operator per face, ...
\[ B_p := \prod_{i \in p} \sigma_i^z \in \mathcal{P}(\mathcal{L}) \]
... and the stabilizer group:
\[ S := \text{span} \{ A_s, B_p \}_{s \in \mathcal{V}(\mathcal{L}), p \in \mathcal{P}(\mathcal{L})} \]
The Stabilizer
Is this a stabilizer?

All star- and plaquette operators commute:

\[
[A_s, A_{s'}] = [B_p, B_{p'}] = [A_s, B_p] = 0
\]

Yes!
The Stabilizer

Dimension of stabilized subspace?
Two non-trivial relations:
Therefore the dimension is:

Encode two logical qubits in $\mathcal{PS}$!
Dual Lattice & Strings

Construction
1. Faces of $\mathcal{L}$ become vertices of $\mathcal{L}^*$
2. Vertices of $\mathcal{L}$ become faces of $\mathcal{L}^*$
2. Edges of $\mathcal{L}$ remain edges of $\mathcal{L}^*$
**Construction**

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**Strings & Loops**

$\mathcal{C} = \text{Open string on } \mathcal{L}$

$\mathcal{C}^* = \text{Open dual string on } \mathcal{L}^*$

Closed strings = Loops
String & Loop operators
String & Loop operators

Representation on $\mathcal{H}(\mathcal{L})$:

$$X[\mathcal{C}^*] := \prod_{i^* \in \mathcal{C}^*} \sigma_{i^*}^x, \quad Z[\mathcal{C}] := \prod_{i \in \mathcal{C}} \sigma_i^z$$
String & Loop operators

Representation on $\mathcal{H}(\mathcal{L})$:

$$X[^*C] := \prod_{i^* \in ^*C} \sigma_{i^*}^x, \quad Z[C] := \prod_{i \in C} \sigma_i^z$$

Commutation relations

Stabilizers commute with strings except for their endpoints!

$$[A_s, X[^*C]] = [B_p, X[^*C]] = 0$$
$$[A_s, Z[C]] = [B_p, Z[C]] = 0$$

→ Loop operators commute with all stabilizers!

→ Loop operators are diagonalizable over $\mathcal{PS}$
The Code Space
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**Trivial Loops:**

→ Boundaries of plaquettes/stars
→ Act as *identity* on $\mathcal{PS}$
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**Non-Trivial Loops:**
- Wrap around the torus on $\mathcal{L}$ or $\mathcal{L}^*$
- Four such operators: $X_1, X_2, Z_1, Z_2$
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Algebra

\[
[Z_i, X_j] = [X_i, X_j] = [Z_i, Z_j] = 0
\]

\[
\{Z_i, X_i\} = 0, \quad i \neq j
\]

→ Pauli algebra of 2 qubits on $\mathcal{P}S$
The Code Space

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→ $Z_1$ and $Z_2$ commute

Eigenbasis

$$Z_j |v_1, v_2\rangle = v_j |v_1, v_2\rangle, \quad v_j \in \{-1, 1\}$$
The Code Space

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**Eigenbasis**

$$Z_j |v_1, v_2\rangle = v_j |v_1, v_2\rangle, \quad v_j \in \{-1, 1\}$$

- Action of $X_1$ and $X_2$:

$$X_j |v_1, v_2\rangle = \begin{cases} |-v_1, v_2\rangle, & j = 1 \\ |v_1, -v_2\rangle, & j = 2 \end{cases}$$
Error detection

1. Initial state
   → System in the code space $\mathcal{PS}$
   → Here: $|v_1, v_2\rangle = |1, 1\rangle$
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→ Here: \( |v_1, v_2\rangle = |1, 1\rangle \)

2. Errors occur
→ Independent single-qubit errors:

\[
\begin{align*}
\text{Spin-Flip:} & \quad \mathcal{PS} \ni |\Psi\rangle \rightarrow \sigma_x^i |\Psi\rangle \notin \mathcal{PS} \\
\text{Phase-Flip:} & \quad \mathcal{PS} \ni |\Psi\rangle \rightarrow \sigma_z^i |\Psi\rangle \notin \mathcal{PS}
\end{align*}
\]
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   → Errors = (open) String operators
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3. Syndrome measurement
   → Measure all star/plaquette operators:

   - Spin-Flip: \( B_p \sigma^x_i |\Psi\rangle = -\sigma^x_i |\Psi\rangle \quad i \in p \)
   - Phase-Flip: \( A_s \sigma^z_i |\Psi\rangle = -\sigma^z_i |\Psi\rangle \quad i \in s \)

   → Syndrome = Endpoints of error strings
Error correction
4. Computation of correction

→ Minimum weight perfect matching:
  Pair & connect endpoints with shortest total path length
Error correction

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→ Minimum weight perfect matching:
   Pair & connect endpoints with shortest total path length

5. Application of correction
→ Error strings + correction operators:

\[ X[C_{pq}^*]X[J_{pq}^*] = 1 \quad Z[C_{rs}]Z[J_{rs}] = Z_2 \]
Error correction

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5. Application of correction
→ Error strings + correction operators:

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X[C_{pq}^*] X[J_{pq}^*] = 1 \quad \text{successful correction}
\]
\[
Z[C_{rs}] Z[J_{rs}] = Z_2 \quad \text{failed correction}
\] (logical phase error)
Error correction

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→ Minimum weight perfect matching:
   Pair & connect endpoints with shortest total path length

5. Application of correction
→ Error strings + correction operators:

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\[ Z[C_{rs}]Z[J_{rs}] = Z_2 \quad \text{failed correction} \text{ (logical phase error)} \]

Code distance
→ Correction without fail if less than
\[ \left\lfloor \frac{L-1}{2} \right\rfloor \]
grows with L!

errors of the same type occurred.
Experimental results

Revealing anyonic features in a toric code quantum simulation

Experimental demonstration of topological error correction

State preservation by repetitive error detection in a superconducting quantum circuit

Superconducting nanocircuits for topologically protected qubits

→ NV-centers still missing ...

State based
(Photons, Josephson junctions)

Hamiltonian based
(Josephson junctions)
The End. Questions?